Closed book, closed notes, calculators OK. The Signal Processing problem consists of two parts, each comprising multiple questions, points as marked (point sum = 40 = perfect score); nominal duration = 1 hour.

Part A: Filtering, (down-)sampling, and the roots of unity [sum of points = 25]

- [**3 points**] For $a \in \mathbb{C}$, define $s(N) := 1 + a + a^2 + \cdots + a^{N-1}$. Express $s(N+1)$ as a function of $s(N)$ *in two different ways*, and use the resulting expressions to derive a closed-form expression for $s(N)$.

- [**5 points**] Consider a discrete-time filter with impulse response

$$h(k) = \begin{cases} 1, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

where k is the discrete time variable, and N is a non-negative integer constant (the length of the impulse response). Compute the discrete-time Fourier transform $H(e^{j\omega}) := \sum_{k=-\infty}^{+\infty} h(k)e^{-j\omega k}$.

- [**5 points**] Suppose that a discrete-time signal $x(k)$ is input to the above filter, and the output $y(k) = 0$, $\forall k$. What can you say about (i.e., can you *characterize*) the input signal $x(k)$ in this case? Be as specific as you can.

- [**5 points**] Now assume that the discrete-time Fourier transform $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$ of $x(k)$ satisfies $X(e^{j\omega}) = 0$, $\forall |\omega| > \omega_c$. Find a condition on ω_c that allows recovery of the input signal $\{x(k)\}_{k=-\infty}^{\infty}$ from the output signal $\{y(k)\}_{k=-\infty}^{\infty}$.

- **[2 points]** Suppose you down-sample the filter's output $\{y(k)\}_{k=-\infty}^{\infty}$ by a factor of N, and let $\{v(k)\}_{k=-\infty}^{\infty}$ with $v(k) := y(Nk)$, $\forall k \in \mathbb{Z}$, be the down-sampled signal. Under what condition can you recover the signal $\{y(k)\}_{k=-\infty}^{\infty}$ from the signal $\{v(k)\}_{k=-\infty}^{\infty}$? (no need to prove the (down-)sampling theorem here, simply state the condition).

- **[5 points]** Write out an expression for $v(k)$ as a function of the filter's *input* signal. Combining the above results, state a condition under which you can exactly recover the filter's input signal $\{x(k)\}_{k=-\infty}^{\infty}$, from the down-sampled version $\{v(k)\}_{k=-\infty}^{\infty}$ of the filter's output signal. Comment on the result.

Part B: Phase retrieval? [sum of points = 15]

- **[5 points]** Show by simple counter-example that it is generally impossible to recover a discrete-time signal $\{x(k)\}_{k=-\infty}^{\infty}$ from only the magnitude $|X(e^{j\omega})|$ of its discrete-time Fourier transform $X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$. I.e., show that we need the phase of $X(e^{j\omega})$ too, $|X(e^{j\omega})|$ is not enough.

- **[10 points]** Describe how you can construct a non-trivial class of signals $\{x(k)\}_{k=-\infty}^{\infty}$ for which perfect recovery of $\{x(k)\}_{k=-\infty}^{\infty}$ is possible from $|X(e^{j\omega})|$ only. Hint: How can you generate $X(e^{j\omega}) \geq 0$ (non-negative real) $\forall\omega$?